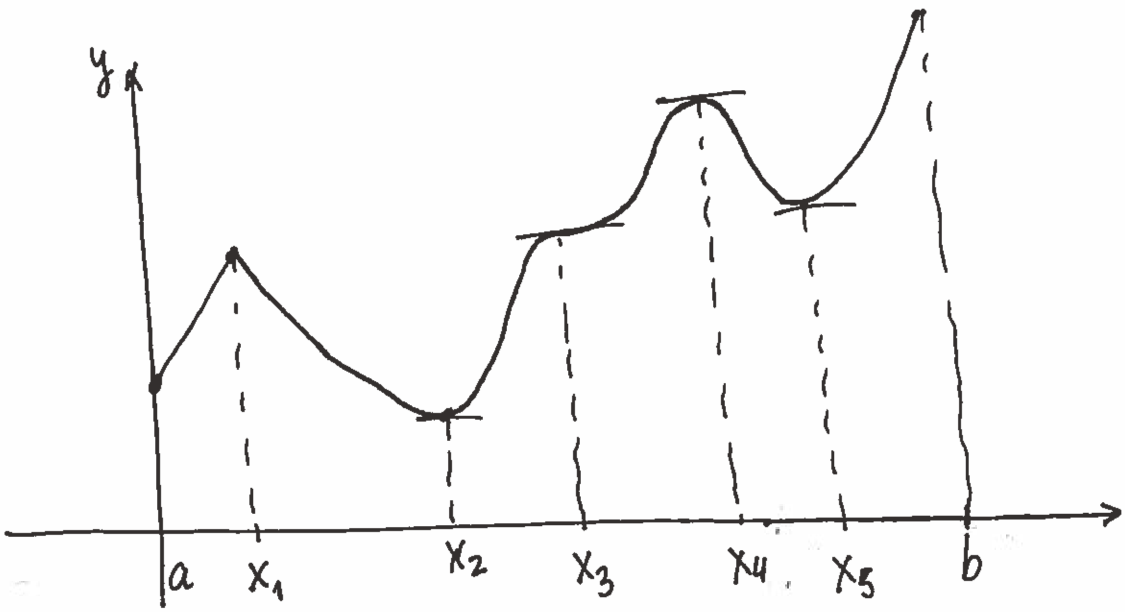
14.7 (Cont.)

MAXIMUM AND MINIMUM ON BOUNDED AND CLOSED DOMAINS

Closed Interval Method.

1-D



f is continuous on $[a,b]$.

$f'(x_1)$ DNE. , f is differentiable on (a,b) except at $x = x_1$.

To find abs. max. and abs. minimum

Evaluate $f(x_i)$, $i=1,..,5$, $f(a)$ and $f(b)$ and compare.
↳ critical points

Largest is abs. max. : $f(b)$

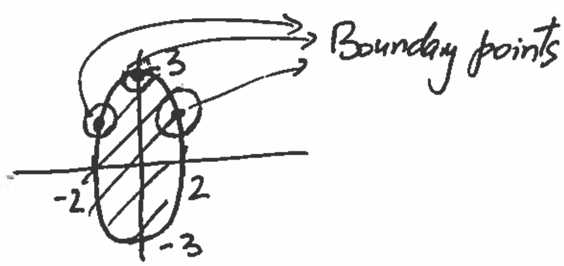
Smallest is abs. min. : $f(x_2)$.

14.7 Absolute Maximum and minimum Values.

Def.- A **boundary point of D** is a point (a,b) such that every disk $B_r(a,b)$ contains points in D and not in D.

For example,

If $D: \frac{x^2}{4}+\frac{y^2}{9}\leq 1$

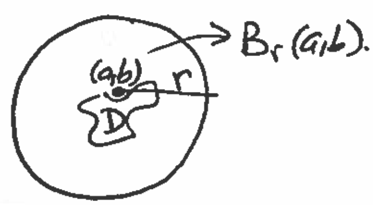


Def.- A closed set H is one that contains **ALL** its boundary points.

$$\frac{x^2}{4}+\frac{y^2}{9}\leq 1 \text{ is closed}$$

but $$\frac{x^2}{4}+\frac{y^2}{9}< 1 \text{ is not closed,}$$

Def.- A bounded set is one that is contained within some disk.



Show thm and procedure (LaTex document).

Exercise # 37

Find the absolute max and min. of

$z = f(x,y) = 2x^3 + y^4$ on the set $D = \{(x,y) \mid x^2+y^2 \leq 1\}$.

answ:

i) Find critical points:

$$f_x(x,y) = 6x^2 = 0, \qquad f_y(x,y) = 4y^3 = 0$$

then $(0,0)$ only critical point and $f(0,0) = 0$

ii) Find extreme values of f on $\mathcal{C}$ boundary of D.

$$x^2+y^2=1, \text{ then } \quad y^2 = 1-x^2, \quad -1 \leq x \leq 1.$$

Thus, along the boundary f taces the values:

$$\tilde{f}(x) = 2x^3 + (1-x^2)^2 = 2x^3+1-2x^2+x^4, \text{ and } -1 \leq x \leq 1$$

This function describes the values of f along the boundary Circle $\mathcal{C}$. To obtain the points where $\tilde{f}$ reach max and min, we proceed as we did in Calculus of 1 Variable

Critical points: $\tilde{f}'(x) = 6x^2 - 4x + 4x^3 = 0$

or $$2x(2x^2+3x-2)=0$$

roots are $\boxed{x=0}$ and

$$x=\frac{-3\pm\sqrt{9+16}}{4}=\frac{-3\pm5}{4}\begin{cases}\boxed{1/2}\\ \boxed{-2}\end{cases}$$

The point $x=-2$ is not in the domain of $\tilde{f}$ $(-1\le x\le 1)$

So the only candidates of D for absolute values are

$$x=0,\quad y^2=1 \Rightarrow y=\pm1$$
$$x=1/2,\quad y^2=1-1/4=3/4 \Rightarrow y=\pm\frac{\sqrt{3}}{2}$$

or boundary points where a extreme value can be reached are:

$$(0,1),\ (0,-1),\ (1/2, \sqrt{3}/2),\ (1/2, -\sqrt{3}/2).$$

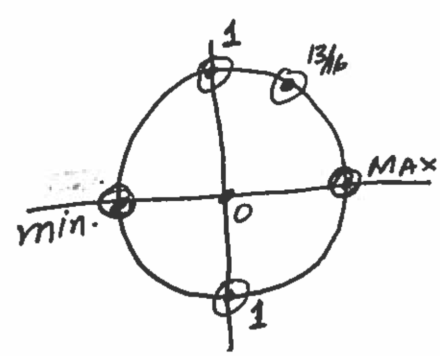
But, we also need to include the end points along boundary 6 when $x=\pm1$.

or $$(1,0) \text{ and } (-1,0).$$

Evaluating f at critical and boundary points:

$f(0,0)=0$, $f(0,1)=1$, $f(0,-1)=1$, $f(1/2,\sqrt{3}/2)=\frac{13}{16}$

$f(-1,0)=-2$, $f(1,0)=2$.

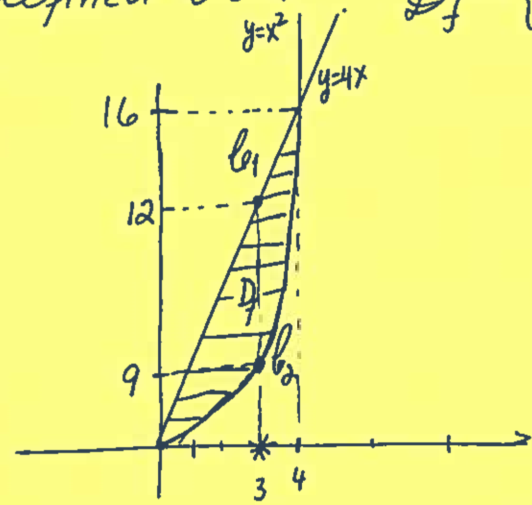


14.7.

Example: Consider the function

$$f(x,y) = (x-3)^2 + y^2$$

defined over $\mathcal{D}_f = \{(x,y) : 0 \le x \le 4, \quad x^2 \le y \le 4x\}$



$\mathcal{C}_1 : \hat{r}_1(t) = (t, 4t), \quad t \in [0,4]$.

$\mathcal{C}_2 : \hat{r}_2(t) = (t, t^2), \quad t \in [0,4]$.

Find global max and global minimum for $f(x,y)$.

The function $f(x,y)$ is differentiable at every interior point in domain $\mathcal{D}_f$. Therefore, the only critical points are either Stationary points or boundary points.

Stationary points:

$$\nabla f(x,y) = (2(x-3), 2y) = 0 \quad \text{iff} \quad \begin{matrix} x = 3 \\ y = 0 \end{matrix}$$

But this point $(3,0) \notin \mathcal{D}_f$. Therefore, the only possibility for global max or global min is over the boundary $\mathcal{C}_1 \cup \mathcal{C}_2$.

Global max and Global min over boundary $\mathcal{C}_1$

$$f_1(t) = f(\vec{r}_1(t)) = f(t, 4t) = (t-3)^2 + 16t^2 = 17t^2 - 6t + 9, \quad t \in [0,4]$$

$$\Rightarrow f_1'(t) = 34t - 6 = 0 \Rightarrow t = \frac{6}{34} = \frac{3}{17}$$

Thus, $t^* = \frac{3}{17}$ is an stationary point over the boundary $\mathcal{C}_1$ (straight line). → The corresponding point in D for $t^*=3/17$ is $(3/17, 12/17)$

Also, $f_1''(t) = 34 > 0 \Rightarrow t^* = \frac{3}{17}$ is a local minimum

and $f_1(3/17) = f(3/17, 12/17) \simeq 8.47$.

To find global minimum and maximum, we need to consider also the extreme of the intervals $[0,4]$

In fact,

$$f_1(0) = f(0,0) = 9$$

$$f_1(4) = f(4,16) = 1^2 + 16^2 = 257.$$

Therefore, Global maximum over $\mathcal{C}_1$: at $(4,16)$

$$f(4,16) = 257.$$

Global minimum over $\mathcal{C}_1$: at $(3/17, 12/17)$

$$f(3/17, 12/17) \simeq 8.47.$$

<u>Global maximum and global minimum over boundary $\mathcal{C}_2$</u>

$$f_2(t) = f(\vec{r}_2(t)) = f(t, t^2) = (t-3)^2 + t^4 = t^4 + t^2 - 6t + 9, \quad t \in [0,4]$$

$$\Rightarrow f_2'(t) = 4t^3 + 2t - 6 = 0 \Leftrightarrow (t-1)(4t^2+4t+6) = 0$$

$t=1$ only real root.

Since, $f_2''(t) = 12t + 2\Big|_{t=1} = 14 > 0$

then $\hat{t} = 1$ is a local minimum

and $f_2(1) = f(1,1) = 4 + 1 = 5$

To find global max and global min over $\mathcal{C}_2$, we need to consider also the ends of the interval $[0,4]$.

In fact,

$$f_2(0) = f(0,0) = 9$$

$$f_2(4) = f(4,16) = 1 + 16^2 = 257$$

Therefore,

Global maximum over $\mathcal{C}_2$: at $(4,16)$, $f(4,16) = 257$

Global minimum over $\mathcal{C}_2$: at $(1,1)$, $f(1,1) = 5$

Finally, Global maximum of $f(x,y)$ over D_f: at $(4,16)$, $f(4,16) = 257$.

Global minimum of $f(x,y)$ over D_f: at $(1,1)$, $f(1,1) = 5$.